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Osaka J. Math.
44 (2007), 765–793

HERMITIAN STRUCTURES ON COTANGENT BUNDLES OF FOUR DIMENSIONAL SOLVABLE LIE GROUPS

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(Received June 13, 2006, revised November 8, 2006)

Abstract

We study hermitian structures, with respect to the standard neutral metric on the cotangent bundle T^*G of a 2n-dimensional Lie group G, which are left invariant with respect to the Lie group structure on T^*G induced by the coadjoint action. These are in one-to-one correspondence with left invariant generalized complex structures on G. Using this correspondence and results of [8] and [10], it turns out that when G is nilpotent and four or six dimensional, the cotangent bundle T^*G always has a hermitian structure. However, we prove that if G is a four dimensional solvable Lie group admitting neither complex nor symplectic structures, then T^*G has no hermitian structure or, equivalently, G has no left invariant generalized complex structure.

1. Introduction

The cotangent bundle T^*G of a Lie group G with Lie algebra $\mathfrak{g}$ has a canonical Lie group structure induced by the coadjoint action of G on $\mathfrak{g}^*$ and also a canonical bi-invariant neutral metric. With respect to this data, hermitian structures on T^*G such that left translations are holomorphic isometries are given by endomorphisms J of $\mathfrak{g} \oplus \mathfrak{g}^*$ satisfying $J^2 = -\operatorname{Id}$ which are orthogonal with respect to

$$(1) \qquad \langle (x, \alpha), (y, \beta) \rangle = \frac{1}{2}(\beta(x) + \alpha(y)),$$

and satisfy $N_J \equiv 0$, where N_J is defined in (3), with respect to the Lie bracket:

$$(2) \qquad [(x, \alpha), (y, \beta)] = ([x, y], -\beta \circ \operatorname{ad}(x) + \alpha \circ \operatorname{ad}(y)) \quad \text{for} \quad x, y \in \mathfrak{g}, \ \alpha, \beta \in \mathfrak{g}^*.$$

On the other hand, $\mathfrak{g} \oplus \mathfrak{g}^*$ is the fiber at the identity e of the bundle $TG \oplus T^*G$ over G and one may extend J above to the whole $TG \oplus T^*G$ using the standard lift of left multiplication in G. The Courant bracket (see (12) below), when restricted to left invariant vector fields and left invariant 1-forms is given by the equation above thus

2000 Mathematics Subject Classification. Primary 17B30, 53C15, 22E25; Secondary 53C55, 53D17.

establishing a correspondence, in the invariant case, between invariant hermitian structures on T^*G and left invariant generalized complex structures on G (Proposition 3.1). It follows that any such structure gives rise to a Poisson Lie group structure on T^*G such that the dual Poisson Lie group $(T^*G)^*$ is a complex Lie group (Corollary 3.2).

The concept of *generalized complex structure* was introduced by Hitchin [12] and developed by Gualtieri [11]. Symplectic and complex geometry are extremal special cases of generalized complex geometry. In [8] Cavalcanti and Gualtieri show that the 34 classes of 6-dimensional nilpotent Lie groups (see [14, 20] for the classification) have a left invariant generalized complex structure; but, five of these classes of nilpotent Lie groups admit neither symplectic nor complex left invariant geometries (see [20]). It is proved in [10] that every four dimensional nilpotent Lie group has left invariant symplectic structures and hence generalized complex structures. So, it seems interesting to understand the way this property occurs on non-nilpotent solvable Lie groups.

In this paper we deal with left invariant generalized complex structures on solvable Lie groups of dimension 4. To this end, in Proposition 3.1 of §3, we show that there is a one-to-one correspondence between left invariant generalized complex structures on a Lie group G and invariant hermitian structures (J, g) on T^*G, where g is the standard neutral metric on T^*G. In §4 we prove Theorem 4.7 which asserts that *a four dimensional solvable Lie group G has neither left invariant symplectic nor complex structures if and only if G does not admit generalized complex structures*. In the proof, we use the classification of 4-dimensional solvable Lie groups with left invariant complex (resp. symplectic) structures carried out in [23] and [17] (resp. [15]; see also [18]).

On the other hand, in §5 we distinguish the solvable Lie groups of dimension 4 admitting a non-extremal left invariant generalized complex structure (§5.1) and the Lie groups carrying a left invariant complex or symplectic structure but without a non-extremal left invariant generalized complex structure (§5.2).

Finally, in §6 we show that Theorem 4.7 does not work in dimension 6. In fact, we construct an example of a six dimensional (non-nilpotent) solvable Lie group admitting neither left invariant symplectic nor complex structures but having non-extremal generalized complex structures.

2. Hermitian structures on cotangent Lie groups

A *left invariant complex structure* on a real Lie group G is a complex structure on the underlying manifold such that left multiplication by elements of the group are holomorphic. Equivalently, there exists an endomorphism J of $\mathfrak{g}$, the Lie algebra of G, such that: $J^2 = -\,\mathrm{Id}$ and $N_J \equiv 0$, where

$$N_J(x, y) = [x, y] + J[Jx, y] + J[x, Jy] - [Jx, Jy], \quad \forall\, x, y \in \mathfrak{g}. \tag{3}$$

The condition $N_J \equiv 0$ is called the *integrability condition* of J.

The action of G on itself given by left multiplication $L_g\colon G \to G$ can be lifted to an action of G on TG given by $dL_g\colon TG \to TG$. Thus, a left invariant complex structure is an equivariant endomorphism of TG with respect to the lifted action of G given by left multiplication. Similarly, a left invariant symplectic structure on G is an equivariant isomorphism $\omega\colon TG \to T^*G$ where the action of G on T^*G is $L^*_{g^{-1}}\colon T^*G \to T^*G$.

A *left invariant hermitian structure* on G is a pair (J, g) of a left invariant complex structure J together with a left invariant hermitian metric g (not necessarily positive definite). If J denotes the corresponding endomorphism on $\mathfrak{g}$ and $\langle \cdot\,, \cdot \rangle$ the non degenerate symmetric bilinear form on $\mathfrak{g}$ induced by g, we say that $(J, \langle \cdot\,, \cdot \rangle)$ is a hermitian structure on $\mathfrak{g}$. A non degenerate symmetric bilinear form $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{g}$ is said to be ad-invariant when it satisfies:

$$\langle [x, y], z\rangle + \langle y, [x, z]\rangle = 0 \quad \text{for any} \quad x, y, z \in \mathfrak{g}. \tag{4}$$

If G is a Lie group with Lie algebra $\mathfrak{g}$ and g is a bi-invariant metric on G, that is, g is both left and right invariant, then the bilinear form $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{g}$ induced by g is ad-invariant.

Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the real Lie algebra $\mathfrak{g}$ and let σ denote the conjugation in $\mathfrak{g}^{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$, that is, $\sigma(x + iy) = x - iy$, $x, y \in \mathfrak{g}$. Starting with a hermitian structure $(J, \langle \cdot\,, \cdot \rangle)$ on $\mathfrak{g}$, let $J^{\mathbb{C}}$ (resp. $\langle \cdot\,, \cdot \rangle^{\mathbb{C}}$) denote the complex linear (resp. complex bilinear) extension of J (resp. $\langle \cdot\,, \cdot \rangle$) to $\mathfrak{g}^{\mathbb{C}}$. We obtain a splitting

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{q} \oplus \sigma(\mathfrak{q}),$$

where $\mathfrak{q}$, the i-eigenspace of $J^{\mathbb{C}}$, is a complex subalgebra of $\mathfrak{g}^{\mathbb{C}}$ which is maximal isotropic with respect to $\langle \cdot\,, \cdot \rangle^{\mathbb{C}}$.

We prove the above statement in the following proposition, where it is shown that, conversely, if B is a symmetric bilinear form on $\mathfrak{g}^{\mathbb{C}}$ satisfying certain conditions, then any splitting $\mathfrak{g}^{\mathbb{C}} = \mathfrak{q} \oplus \sigma(\mathfrak{q})$, where $\mathfrak{q}$ is a maximal B-isotropic complex subalgebra of $\mathfrak{g}^{\mathbb{C}}$, gives rise to a hermitian structure $(J, \langle \cdot\,, \cdot \rangle)$ on $\mathfrak{g}$ such that the i-eigenspace of $J^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}$ is $\mathfrak{q}$ and $\langle x, y\rangle = B(x, y)$ for $x, y \in \mathfrak{g}$.

Proposition 2.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and denote by $\mathfrak{g}^{\mathbb{C}}$ the complexification of $\mathfrak{g}$. There is a one-to-one correspondence between left invariant hermitian structures (J, g) on G and pairs $(\mathfrak{q}, B)$, where B is a symmetric bilinear form on $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{q}$ is a maximal B-isotropic complex subalgebra of $\mathfrak{g}^{\mathbb{C}}$ satisfying the following conditions*:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{q} \oplus \sigma(\mathfrak{q}), \tag{5}$$

$$B(\sigma z, \sigma w) = \overline{B(z, w)}, \quad z, w \in \mathfrak{g}^{\mathbb{C}}, \tag{6}$$

where $\overline{\alpha}$ *denotes the complex conjugate of* $\alpha \in \mathbb{C}$ *and* σ *is the conjugation in* $\mathfrak{g}^{\mathbb{C}}$ *with respect to* $\mathfrak{g}$.

Proof. Given a left invariant hermitian structure (J, g) on G, let $(J, \langle\,\cdot\,,\,\cdot\,\rangle)$ be the corresponding hermitian structure on $\mathfrak{g}$. $\mathfrak{g}^{\mathbb{C}}$ decomposes into a direct sum of subspaces $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$, the eigenspaces of $J^{\mathbb{C}}$ of eigenvalue i and $-i$, respectively. It follows that

$$\mathfrak{g}^{1,0} = \{x - iJx : x \in \mathfrak{g}\}, \quad \mathfrak{g}^{0,1} = \{x + iJx : x \in \mathfrak{g}\},$$

hence, $\mathfrak{g}^{0,1} = \sigma(\mathfrak{g}^{1,0})$. Equation $N_J \equiv 0$ is equivalent to the fact that these subspaces are subalgebras. Moreover, using that J is orthogonal, it is easy to check that both $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ are isotropic with respect to $\langle\,\cdot\,,\,\cdot\,\rangle^{\mathbb{C}}$. Since $\langle\,\cdot\,,\,\cdot\,\rangle$ is non degenerate, these subalgebras are maximal isotropic. Hence, $(\mathfrak{g}^{0,1}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathbb{C}})$ satisfies the required conditions. Note that equation (6) holds if and only if B takes real values on $\mathfrak{g}$, and $\langle\,\cdot\,,\,\cdot\,\rangle^{\mathbb{C}}$ clearly satisfies this property.

Conversely, given a pair $(\mathfrak{q}, B)$ as in the statement, we wish to show that it gives rise to a hermitian structure $(J, \langle\,\cdot\,,\,\cdot\,\rangle)$ on $\mathfrak{g}$. Let J be the almost complex structure defined on $\mathfrak{g}^{\mathbb{C}}$ by

$$Jz = iz, \quad J \circ \sigma(z) = -i\sigma(z), \quad z \in \mathfrak{q}.$$

Since $J \circ \sigma = \sigma \circ J$, then J leaves $\mathfrak{g}$ stable. The fact that $\mathfrak{q}$ is a subalgebra implies that J satisfies $N_J \equiv 0$. Since equation (6) holds, B takes real values on $\mathfrak{g}$. Let $\langle\cdot,\cdot\rangle$ be the restriction of B to $\mathfrak{g}$. It follows from (5) and the fact that $\mathfrak{q}$ is B-isotropic that J is orthogonal with respect to $\langle\cdot,\cdot\rangle$. Since $\mathfrak{q}$ is maximal isotropic then $\langle\cdot,\cdot\rangle$ is non degenerate, that is, $(J, \langle\cdot,\cdot\rangle)$ is a hermitian structure on $\mathfrak{g}$. Therefore, it induces, by left translations, a left invariant hermitian structure on G and the proposition follows. $\square$

We will be studying a special class of left invariant hermitian structures. The Lie groups that come into the picture are the cotangent bundles of Lie groups with a standard bi-invariant metric.

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{v}$ a $\mathfrak{g}$-module, that is, there exists a Lie algebra homomorphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{v})$. Let $\mathfrak{g} \ltimes_\rho \mathfrak{v}$ denote the semidirect product of $\mathfrak{g}$ by $\mathfrak{v}$, where we look upon $\mathfrak{v}$ as an abelian Lie algebra. The bracket on $\mathfrak{g} \ltimes_\rho \mathfrak{v}$ is given as follows:

$$[(x, u), (y, v)] = ([x, y], \rho(x)v - \rho(y)u) \quad \text{for} \quad x, y \in \mathfrak{g},\ u, v \in \mathfrak{v}. \tag{7}$$

Complex structures on Lie algebras of the above type were studied in [4]. In the present article we will restrict our attention to the particular case when $\mathfrak{v} = \mathfrak{g}^*$ and $\rho = \mathrm{ad}^*$ is the coadjoint representation:

$$\mathrm{ad}^*(x)(\alpha) = -\alpha \circ \mathrm{ad}(x), \quad \alpha \in \mathfrak{g}^*,\ x \in \mathfrak{g}.$$

We will denote $\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*$ by $(T^*\mathfrak{g}, \mathrm{ad}^*)$, the Lie bracket being given by (2). The cotangent algebra $(T^*\mathfrak{g}, \mathrm{ad}^*)$ has a standard non degenerate symmetric ad-invariant bilinear form $\langle \cdot\,, \,\cdot \rangle$ (see (1)). We notice that the subalgebra $\mathfrak{g}$ and the ideal $\mathfrak{g}^*$ are maximal isotropic in $(T^*\mathfrak{g}, \langle \cdot\,, \,\cdot \rangle)$.

Left invariant hermitian structures on the cotangent Lie group T^*G are given by endomorphisms J of $T^*\mathfrak{g}$ whose matrix form with respect to the decomposition $\mathfrak{g} \oplus \mathfrak{g}^*$ is

$$J = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix},$$

and satisfy

$$\text{(8)} \qquad \begin{aligned} &\text{(i)} && J_4 = -J_1^*, \quad J_2 = -J_2^*, \quad J_3 = -J_3^*, \\ &\text{(ii)} && J_1^2 + J_2 J_3 = -\mathrm{Id}\,, \quad J_1 J_2 = -(J_1 J_2)^*, \quad J_3 J_1 = -(J_3 J_1)^*, \\ &\text{(iii)} && J \quad \text{is integrable.} \end{aligned}$$

EXAMPLE 2.2. Let $\jmath$ be a complex structure on $\mathfrak{g}$, $\dim \mathfrak{g} = 2n$, and define $J_\jmath$ on $T^*\mathfrak{g}$ by

$$\text{(9)} \qquad J_\jmath(x, \alpha) = (\jmath(x), -\jmath^*(\alpha)), \quad x \in \mathfrak{g},\ \alpha \in \mathfrak{g}^*,$$

where $\jmath^*$ is the adjoint of $\jmath$, that is, $\jmath^*(\alpha) = \alpha \circ \jmath$. It follows that $J_\jmath$ is orthogonal with respect to the standard bilinear form $\langle \cdot\,, \,\cdot \rangle$ on $T^*\mathfrak{g}$. Moreover, it was shown in [4] (Proposition 3.2) that the integrability of $\jmath$ implies that $J_\jmath$ is a complex structure on $(T^*\mathfrak{g}, \mathrm{ad}^*)$. Therefore, $(J_\jmath, \langle \cdot\,, \,\cdot \rangle)$ is a hermitian structure on $(T^*\mathfrak{g}, \mathrm{ad}^*)$.

EXAMPLE 2.3. Let $\omega\colon \mathfrak{g} \to \mathfrak{g}^*$ be a linear isomorphism and define

$$\text{(10)} \qquad J_\omega(x, \alpha) = (-\omega^{-1}(\alpha), \omega(x)),$$

(compare with §4 in [4]). It follows that J_ω is orthogonal with respect to the standard bilinear form on $T^*\mathfrak{g}$ if and only if ω is skew-symmetric. The integrability of J_ω is equivalent to the following condition

$$\text{(11)} \qquad \omega([x, y]) = \omega(x) \circ \mathrm{ad}(y) - \omega(y) \circ \mathrm{ad}(x).$$

Therefore, if ω satisfies (11) J_ω defines a hermitian structure on $(T^*\mathfrak{g}, \mathrm{ad}^*)$. We observe that in this case, ω is a symplectic structure on $\mathfrak{g}$.

3. Left invariant generalized complex structures on Lie groups

We recall that a generalized complex structure on a manifold M is an endomorphism $\mathcal{J}$ of $TM \oplus T^*M$ satisfying $\mathcal{J}^2 = -\mathrm{Id}$ which is orthogonal with respect to the standard inner product $\langle \cdot, \cdot \rangle$ on $TM \oplus T^*M$ defined in (1) and such that the i eigenbundle of $\mathcal{J}$ in $(TM \oplus T^*M) \otimes \mathbb{C}$ is involutive with respect to the Courant bracket. This bracket is defined as follows:

$$[(X, \xi), (Y, \eta)] = \left([X, Y], \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi) \right), \tag{12}$$

where (X, ξ), (Y, η) are smooth sections of $TM \oplus T^*M$.

When M is a Lie group G, consider the left action of G on $TG \oplus T^*G$ induced by left multiplication of G on itself, that is,

$$\begin{aligned} &\lambda \colon G \times (TG \oplus T^*G) \to TG \oplus T^*G, \\ &\qquad (g, (x, \alpha)) \mapsto (dL_g)_h x, (L^*_{g^{-1}})_{gh} \alpha), \quad x \in T_h G,\ \alpha \in T^*_h G,\ g, h \in G \end{aligned} \tag{13}$$

where

$$\left(L^*_{g^{-1}}\right)_{gh} \alpha(y) = \alpha((dL_{g^{-1}})_{gh} y), \quad \forall y \in T_{gh} G.$$

A generalized complex structure $\mathcal{J}$ on G is said to be left invariant (or G-invariant) if

$$\mathcal{J} \colon TG \oplus T^*G \to TG \oplus T^*G$$

is equivariant with respect to the induced left action of G on $TG \oplus T^*G$ given in (13). It follows that, for any $g \in G$, the following diagram is commutative:

$$\begin{tikzcd} T_g G \oplus T^*_g G \arrow[r, "\mathcal{J}_g"] \arrow[d, "\lambda_{g^{-1}}"'] & T_g G \oplus T^*_g G \arrow[d, "\lambda_{g^{-1}}"] \\ \mathfrak{g} \oplus \mathfrak{g}^* \arrow[r, "\mathcal{J}_e"'] & \mathfrak{g} \oplus \mathfrak{g}^* \end{tikzcd},$$

where

$$\lambda_{g^{-1}}(x, \alpha) = \lambda(g^{-1}, (x, \alpha)), \quad x \in T_g G,\ \alpha \in T^*_g G.$$

In other words, $\mathcal{J}$ is left invariant if and only if, for any $g \in G$, $\mathcal{J}_g$ is given in terms of $\mathcal{J}_e$ as follows:

$$\mathcal{J}_g = \lambda_g \circ \mathcal{J}_e \circ \lambda_{g^{-1}} = \begin{pmatrix} (dL_g)_e & \\ & \left(L^*_{g^{-1}}\right)_g \end{pmatrix} \circ \mathcal{J}_e \circ \begin{pmatrix} (dL_{g^{-1}})_g & \\ & \left(L^*_g\right)_e \end{pmatrix}. \tag{14}$$

If we identify the space of left invariant sections of $TG \oplus T^*G$ with $\mathfrak{g} \oplus \mathfrak{g}^*$, then the restriction of the Courant bracket (12) to $\mathfrak{g} \oplus \mathfrak{g}^*$ is precisely the Lie bracket (2) on the cotangent algebra $(T^*\mathfrak{g}, \mathrm{ad}^*)$. Therefore, the Courant integrability condition of a left invariant generalized complex structure $\mathcal{J}$ on G is equivalent to the integrability of $\mathcal{J}_e$ on the cotangent algebra $(T^*\mathfrak{g}, \mathrm{ad}^*)$. Moreover, since λ_g, $g \in G$, are isometries of the standard bilinear form $\langle \cdot\,, \cdot \rangle$ on $TG \oplus T^*G$, it follows that $\mathcal{J}$ is orthogonal with respect to $\langle \cdot\,, \cdot \rangle$ if and only if $\mathcal{J}_e$ is compatible with $\langle \cdot\,, \cdot \rangle$. Therefore, if $\mathcal{J}$ is a left invariant generalized complex structure on G, $(\mathcal{J}_e, \langle \cdot\,, \cdot \rangle)$ is a hermitian structure on $T^*\mathfrak{g}$. Conversely, given a hermitian structure $(J, \langle \cdot\,, \cdot \rangle)$ on $(T^*\mathfrak{g}, \mathrm{ad}^*)$, where $\langle \cdot\,, \cdot \rangle$ is the standard neutral metric on $T^*\mathfrak{g}$, it can be extended, by means of (14), to a left invariant generalized complex structure $\mathcal{J}$ on G such that $\mathcal{J}_e = J$.

The preceding arguments yield the following result:

Proposition 3.1. *There is a one-to-one correspondence between left invariant generalized complex structures on G and invariant hermitian structures (J, g) on T^*G, where g is the standard neutral metric on T^*G.*

When a Lie group G has a left invariant complex or symplectic structure, then any of these structures induces a natural left invariant generalized complex structure on G, as shown in Examples 2.2 and 2.3.

In view of Proposition 3.1, a hermitian structure on $(T^*\mathfrak{g}, \mathrm{ad}^*)$ with respect to the standard bilinear form will be called a generalized complex structure on $\mathfrak{g}$ and denoted by $(\mathcal{J}, \langle \cdot\,, \cdot \rangle)$. When $\mathcal{J}$ satisfies only conditions (i) and (ii) in (8), it will be called an almost generalized complex structure. Note that if $T^*\mathfrak{g}$ is a generalized complex vector space, $\dim \mathfrak{g} = 2n$ (see [11, 6]).

REMARK 1. It was proved in [10] that every four dimensional nilpotent Lie group has either left invariant complex or symplectic structures (maybe both; see also [14] for the classification of these groups). Hence, such a Lie group has a left invariant generalized complex structure. In [8] (see also [7]) it was shown that every six dimensional nilpotent Lie group admits a left invariant generalized complex structure. In other words, the cotangent algebra $(T^*\mathfrak{g}, \mathrm{ad}^*)$ of any four or six dimensional nilpotent Lie algebra $\mathfrak{g}$ admits a hermitian structure $(J, \langle \cdot\,, \cdot \rangle)$, where $\langle \cdot\,, \cdot \rangle$ is the standard bilinear form on $T^*\mathfrak{g}$.

It was proved in [1] that when (J, g) is a left invariant hermitian structure on a Lie group H such that g is bi-invariant then both H and H^* are Poisson Lie groups. Moreover, since J is a complex structure, H^* is a complex Lie group (see [3]). The Poisson structure on H is given by $\Pi(h) = (dR_h)_e J - (dL_h)_e J$, $h \in H$, where J is viewed as an element of $\mathfrak{h} \wedge \mathfrak{h}$ by identifying the Lie algebra $\mathfrak{h}$ of H with its dual $\mathfrak{h}^*$ via the metric g. As a corollary of this result and Proposition 3.1, we therefore obtain:

Corollary 3.2. *If G is a Lie group with a left invariant generalized complex structure, then T^*G and $(T^*G)^*$ are Poisson Lie groups such that $(T^*G)^*$ is a complex Lie group.*

We end this section by determining the generalized complex structures on the two dimensional non-abelian Lie algebra $\mathfrak{g} = \mathfrak{aff}(\mathbb{R})$.

EXAMPLE 3.3. *Generalized complex structures on* $\mathfrak{aff}(\mathbb{R})$. Let $\mathfrak{g} = \mathfrak{aff}(\mathbb{R})$ be the two dimensional non-abelian Lie algebra and $T^*\mathfrak{aff}(\mathbb{R})$ the corresponding cotangent Lie algebra. Let $\{e_0, e_1\}$ be a basis of $\mathfrak{g}$ such that $[e_0, e_1] = e_1$, and $\{\alpha^0, \alpha^1\}$ the dual basis of $\mathfrak{g}^*$. Set

$$X_i = (e_{i-1}, 0), \quad X_{i+2} = (0, \alpha^{i-1}), \quad i = 1, 2,$$

then:

$$[X_1, X_2] = X_2, \quad [X_1, X_4] = -X_4, \quad [X_2, X_4] = X_3.$$

A generalized complex structure $\mathcal{J}$ on $\mathfrak{aff}(\mathbb{R})$ takes the following form in the ordered basis $\{X_1, \ldots, X_4\}$:

$$\mathcal{J} = \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & -a_{14} & 0 \\ 0 & -a_{41} & -a_{11} & -a_{21} \\ a_{41} & 0 & -a_{12} & -a_{22} \end{pmatrix},$$

with $\mathcal{J}^2 = -\operatorname{Id}$ and $N_{\mathcal{J}} \equiv 0$.

In case $a_{14} \neq 0$, the condition $\mathcal{J}^2 = -\operatorname{Id}$ implies $a_{41} \neq 0$, $a_{11}^2 + a_{14}a_{41} = -1$, $a_{11} = a_{22}$ and $a_{12} = 0 = a_{21}$. Hence,

$$\mathcal{J} = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{11} & -a_{14} & 0 \\ 0 & -a_{41} & -a_{11} & 0 \\ a_{41} & 0 & 0 & -a_{11} \end{pmatrix}, \quad a_{14}a_{41} \neq 0, \; a_{11}^2 + a_{14}a_{41} = -1. \tag{15}$$

It follows that $\mathcal{J}$ as above satisfies $N_{\mathcal{J}} \equiv 0$. In particular, if $a_{11} = 0$, $\mathcal{J}$ arises from a symplectic structure on $\mathfrak{aff}(\mathbb{R})$ as in Example 2.3, but for $a_{11} \neq 0$ $\mathcal{J}$ is not induced by a symplectic or complex structure on $\mathfrak{aff}(\mathbb{R})$. However, since $\mathcal{J}$ is of type 0 (see the paragraph next to (20) in §4) it follows from Theorem 4.1 that it is equivalent to a symplectic structure via a B-field transformation.

In case $a_{14} = 0$, the condition $\mathcal{J}^2 = -\operatorname{Id}$ implies

$$a_{41} = 0, \quad a_{11}^2 + a_{12}a_{21} = -1, \quad a_{12}a_{21} \neq 0, \quad a_{11} = -a_{22}.$$

Therefore,

$$\mathcal{J} = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & -a_{11} & 0 & 0 \\ 0 & 0 & -a_{11} & -a_{21} \\ 0 & 0 & -a_{12} & a_{11} \end{pmatrix}, \quad a_{12}a_{21} \neq 0, \; a_{11}^2 + a_{12}a_{21} = -1, \tag{16}$$

and $\mathcal{J}$ satisfies $N_{\mathcal{J}} \equiv 0$. Note that every generalized complex structure in this family arises from a complex structure on $\mathfrak{aff}(\mathbb{R})$ as in Example 2.2.

We observe that $T^*\mathfrak{aff}(\mathbb{R})$ is isomorphic to the Lie algebra $\mathfrak{d}_4$ (see [2]). This is the unique four dimensional solvable Lie algebra admitting a structure of a Manin triple. The above calculations together with Corollary 3.2 imply that the Lie group $\mathcal{D}_4$ with Lie algebra $T^*\mathfrak{aff}(\mathbb{R})$ is a Poisson Lie group such that the Poisson Lie group $\mathcal{D}_4{}^*$ is a complex Lie group.

Fix two generalized complex structures $\mathcal{J}_1$, $\mathcal{J}_2$ on $\mathfrak{g} = \mathfrak{aff}(\mathbb{R})$ as follows:

$$\mathcal{J}_1 = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & -b & 0 \\ 0 & -c & -a & 0 \\ c & 0 & 0 & -a \end{pmatrix}, \; a^2 + bc = -1, \quad \mathcal{J}_2 = \begin{pmatrix} x & y & 0 & 0 \\ z & -x & 0 & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & x \end{pmatrix}, \; x^2 + yz = -1,$$

and consider $G = -\mathcal{J}_1\mathcal{J}_2$. Observe that $\mathcal{J}_1$ and $\mathcal{J}_2$ commute, therefore $G^2 = \mathrm{Id}$. It follows that G defines a positive definite metric on $\mathfrak{g} \oplus \mathfrak{g}^*$ if and only if $cz < 0$. Therefore, when this condition is satisfied, we obtain generalized Kähler structures on $\mathfrak{aff}(\mathbb{R})$ (see [7, 11]).

4. Solvable Lie groups without generalized complex structures

In this section we prove that a four dimensional (non-nilpotent) solvable Lie group has no left invariant generalized complex structures if and only if it admits neither left invariant symplectic nor left invariant complex structures.

We start by fixing some notation. Let $\{\alpha^i\}_{i=0}^3$ be the basis of $\mathfrak{g}^*$ dual to the basis $\{e_i\}_{i=0}^3$ of $\mathfrak{g}$. Define the basis $\{X_i\}_{i=1}^8$ of $T^*\mathfrak{g}$ by

$$X_i = (e_{i-1}, 0) \quad \text{and} \quad X_{i+4} = (0, \alpha^{i-1}), \quad 1 \le i \le 4. \tag{17}$$

Let $\mathcal{J}$ be a linear endomorphism of $T^*\mathfrak{g}$ whose matrix form is

$$\mathcal{J} = \begin{pmatrix} \mathcal{J}_1 & \mathcal{J}_2 \\ \mathcal{J}_3 & \mathcal{J}_4 \end{pmatrix}, \tag{18}$$

with respect to the basis $\{X_i\}_{i=1}^8$ of $T^*\mathfrak{g}$ defined by (17). If $\mathcal{J}$ is orthogonal with respect to the standard bilinear form on $T^*\mathfrak{g}$ then the matrix of $\mathcal{J}$ is of the form

$$\mathcal{J} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & -a_{16} & 0 & a_{27} & a_{28} \\ a_{31} & a_{32} & a_{33} & a_{34} & -a_{17} & -a_{27} & 0 & a_{38} \\ a_{41} & a_{42} & a_{43} & a_{44} & -a_{18} & -a_{28} & -a_{38} & 0 \\ 0 & -a_{61} & -a_{71} & -a_{81} & -a_{11} & -a_{21} & -a_{31} & -a_{41} \\ a_{61} & 0 & -a_{72} & -a_{82} & -a_{12} & -a_{22} & -a_{32} & -a_{42} \\ a_{71} & a_{72} & 0 & -a_{83} & -a_{13} & -a_{23} & -a_{33} & -a_{43} \\ a_{81} & a_{82} & a_{83} & 0 & -a_{14} & -a_{24} & -a_{34} & -a_{44} \end{pmatrix}. \tag{19}$$

Moreover, taking into account (8), if $\mathcal{J}^2 = -\operatorname{Id}$ then the matrix $\mathcal{J}$ has the following property:

$$\text{for every} \quad 1 \le i \le 4 \quad \text{there exists} \quad j \ne i \quad \text{such that} \quad a_{ij} \ne 0. \tag{20}$$

We will say that $\mathcal{J}$ is of *complex type* if $\mathcal{J}_2 = \mathcal{J}_3 = 0$, $\mathcal{J}$ is of *symplectic type* if $\mathcal{J}_1 = \mathcal{J}_4 = 0$, and $\mathcal{J}$ is said to be of *type* k when $\operatorname{rank}(\mathcal{J}_2) = 2(n-k)$, where $\dim \mathfrak{g} = 2n$ (compare with [11]). Observe that if $\mathcal{J}$ is of complex (resp. symplectic) type then it is of type 2 (resp. 0).

We recall a theorem from [11, 8]

Theorem 4.1 ([8], Theorem 1.1; [11], Theorem 4.35). *Any regular point of type k in a generalized complex $2n$-manifold has a neighbourhood which is equivalent, via a diffeomorphism and a B-field transformation, to the product of an open set in $\mathbb{C}^k$ with an open set in the standard symplectic space $\mathbb{R}^{2n-2k}$.*

The previous theorem implies that a $2n$-dimensional Lie algebra admits a generalized complex structure of type 0 (resp. of type n) if and only if it has a symplectic structure (resp., a complex structure).

In order to prove the main result of this section, we recall the definition of the four dimensional solvable Lie algebras admitting neither symplectic nor complex structures (see [21, 22, 15]). They are

(21)

$$\begin{aligned} \mathbb{R} \times \mathfrak{r}_3 &: [e_1, e_2] = e_2, \quad [e_1, e_3] = e_2 + e_3 \\ \mathbb{R} \times \mathfrak{r}_{3,\lambda} &: [e_1, e_2] = e_2, \quad [e_1, e_3] = \lambda e_3, \quad |\lambda| < 1,\ \lambda \ne 0; \\ \mathfrak{r}_4 &: [e_0, e_1] = e_1, \quad [e_0, e_2] = e_1 + e_2, \quad [e_0, e_3] = e_2 + e_3; \\ \mathfrak{r}_{4,\lambda} &: [e_0, e_1] = e_1, \quad [e_0, e_2] = \lambda e_2, \quad [e_0, e_3] = e_2 + \lambda e_3, \quad \lambda \in \mathbb{R},\ \lambda \ne -1, 0, 1; \\ \mathfrak{r}_{4,\mu,\lambda} &: [e_0, e_1] = e_1, \quad [e_0, e_2] = \mu e_2, \quad [e_0, e_3] = \lambda e_3, \\ & \qquad -1 < \mu < \lambda < 1,\ \mu\lambda \ne 0,\ \mu + \lambda \ne 0. \end{aligned}$$

Next, we show that every Lie algebra $\mathfrak{h}$ included in the list (21) has no left invariant generalized complex structures by analyzing each case. To this end, we will prove that any almost complex structure $\mathcal{J}$ on $T^*(\mathfrak{h})$ does not satisfy the integrability condition. This condition is equivalent to the vanishing of the 256 coefficients N_{ij}^k defined by

$$N_{\mathcal{J}}(X_i, X_j) = \sum_{k=1}^{8} N_{ij}^k X_k, \quad 1 \le i < j \le 8,$$

where $N_{\mathcal{J}}$ is the Nijenhuis tensor of $\mathcal{J}$ (see (3)).

Proposition 4.2. *The Lie algebra $\mathbb{R} \times \mathfrak{r}_3$ does not admit generalized complex structures.*

Proof. We consider the basis $\{X_i\}$ for $T^*(\mathbb{R} \times \mathfrak{r}_3)$ defined by (17). Taking into account (12) and the definition of the Lie algebra $\mathbb{R} \times \mathfrak{r}_3$ given in (21), we see that the only non-zero Lie brackets on $T^*(\mathbb{R} \times \mathfrak{r}_3)$ are

$$\begin{aligned} &[X_2, X_3] = X_3, && [X_2, X_4] = X_3 + X_4, && [X_2, X_7] = -X_7 - X_8, \\ &[X_2, X_8] = -X_8, && [X_3, X_7] = X_6, && [X_4, X_7] = X_6 = [X_4, X_8]. \end{aligned}$$

Suppose that $\mathbb{R} \times \mathfrak{r}_3$ has a generalized complex structure, i.e., $T^*(\mathbb{R} \times \mathfrak{r}_3)$ has a hermitian structure $(\mathcal{J}, \langle\cdot,\cdot\rangle)$. Since all the coefficients N_{ij}^k of the Nijenhuis tensor $N_{\mathcal{J}}$ of $\mathcal{J}$ must be zero, we have $0 = N_{78}^2 = {a_{28}}^2$ and $0 = N_{46}^7 = {a_{23}}^2$, and so $a_{28} = a_{23} = 0$.

Let us consider the equation

$$0 = N_{48}^6 = 1 + {a_{44}}^2 + a_{43}(a_{22} + a_{44} + a_{34}) + a_{24}a_{42} - a_{38}a_{83}. \tag{22}$$

Now, (22) and the equations

$$0 = N_{67}^8 = 2a_{24}a_{27}, \quad 0 = N_{68}^6 = -a_{43}a_{27}, \quad 0 = N_{37}^8 = -a_{43}a_{24} - 2a_{83}a_{27},$$

imply that $a_{27} = 0$. Moreover, because

$$0 = N_{78}^1 = -2a_{16}a_{38}, \quad 0 = N_{37}^1 = a_{16}a_{43}, \quad 0 = N_{46}^1 = a_{16}a_{24},$$

we have $a_{16} = 0$ using (22); and because

$$0 = N_{37}^8 = -a_{24}a_{43}, \quad 0 = N_{48}^3 = -2a_{24}a_{38},$$

and

$$0 = N_{27}^7 = -1 - {a_{33}}^2 + a_{43}(a_{22} - a_{33} - a_{34}) - a_{38}a_{83},$$

we see that $a_{24} = 0$. Finally,

$$0 = N^5_{78} = 2a_{21}a_{38}, \quad 0 = N^8_{18} = -a_{21}a_{43}$$

imply $a_{21} = 0$ using (22). So, in the matrix (19) of $\mathcal{J}$, the unique non-zero entry in the 2nd row is a_{22}, which is not possible by (20). This shows that $\mathcal{J}$ cannot be integrable. □

Proposition 4.3. *For $\lambda \neq 0, \pm 1$, the Lie algebra $\mathbb{R} \times \mathfrak{r}_{3,\lambda}$ has no generalized complex structure.*

Proof. Using the basis $\{X_i\}$ for $T^*(\mathbb{R} \times \mathfrak{r}_{3,\lambda})$, given by (17), and the definition of the Lie algebra $\mathbb{R} \times \mathfrak{r}_{3,\lambda}$ stated in (21), the only non-zero Lie brackets on $T^*(\mathbb{R} \times \mathfrak{r}_{3,\lambda})$ are

$$[X_2, X_3] = X_3, \qquad [X_2, X_4] = \lambda X_4, \quad [X_2, X_7] = -X_7$$
$$[X_2, X_8] = -\lambda X_8, \quad [X_3, X_7] = X_6, \qquad [X_4, X_8] = \lambda X_6.$$

Let $(\mathcal{J}, \langle \cdot, \cdot \rangle)$ be a hermitian structure on $T^*(\mathbb{R} \times \mathfrak{r}_{3,\lambda})$. The integrability condition of $\mathcal{J}$ implies that all the coefficients N^k_{ij} of the Nijenhuis tensor of $\mathcal{J}$ are zero. In particular,

$$0 = N^6_{48} = \lambda(1 + a_{44}{}^2 + a_{24}a_{42} + a_{28}a_{82}) + a_{34}a_{43} - a_{38}a_{83}. \tag{23}$$

Now from (23) and the equations

$$0 = N^1_{68} = -\lambda a_{16}a_{28}, \quad 0 = N^2_{45} = \lambda a_{16}a_{24}, \quad 0 = N^2_{57} = a_{16}a_{27},$$
$$0 = N^8_{57} = \lambda a_{14}a_{27} + a_{17}a_{24} + (\lambda - 1)a_{16}a_{34},$$
$$0 = N^4_{57} = -\lambda a_{18}a_{27} + a_{17}a_{28} - (1 + \lambda)a_{16}a_{38},$$

we obtain $a_{16} = 0$. On the other hand,

$$0 = N^4_{67} = (1 - \lambda)a_{27}a_{28}, \qquad 0 = N^2_{47} = -(1 + \lambda)a_{27}a_{24},$$
$$0 = N^7_{78} = (1 + \lambda)a_{23}a_{38} + (\lambda - 1)a_{27}a_{43}, \quad 0 = N^7_{67} = 2a_{27}a_{23},$$
$$0 = N^8_{37} = (1 - \lambda)a_{23}a_{34} - (1 + \lambda)a_{27}a_{83},$$

and (23) imply that $a_{27} = 0$; and from the equations

$$0 = N^7_{78} = (1 + \lambda)a_{23}a_{38}, \quad 0 = N^7_{68} = (1 + \lambda)a_{23}a_{28},$$
$$0 = N^7_{47} = (\lambda - 1)a_{23}a_{34}, \quad 0 = N^7_{46} = (\lambda - 1)a_{23}a_{24}$$

we conclude that $a_{23} = 0$ using again (23). Moreover,

$$0 = N^5_{48} = \lambda a_{21}a_{24}, \quad 0 = N^5_{47} = a_{24}a_{31} + (\lambda - 1)a_{21}a_{34},$$
$$0 = N^5_{68} = \lambda a_{21}a_{28}, \quad 0 = N^5_{34} = a_{24}a_{71} + (1 + \lambda)a_{21}a_{83},$$

imply that $a_{21} = 0$ using again (23). So, according to (20), ${a_{24}}^2 + {a_{28}}^2 \neq 0$. Since $\lambda \neq 0, \pm 1$, the equations

$$0 = N^6_{37} = 2\lambda a_{24} a_{28},$$
$$0 = N^4_{34} = (\lambda - 1) a_{24} a_{43} + (1 + \lambda) a_{28} a_{83},$$
$$0 = N^8_{78} = (\lambda - 1) a_{28} a_{34} + (1 + \lambda) a_{24} a_{38},$$

imply that $a_{34} a_{43} = a_{38} a_{83} = 0$. Therefore,

$$0 = N^6_{37} = 1 + {a_{33}}^2 + \lambda(a_{34} a_{43} - a_{38} a_{83}),$$

which implies that $0 = 1 + {a_{33}}^2$. But this is not possible, and hence $\mathcal{J}$ cannot be integrable on $T^*(\mathbb{R} \times \mathfrak{r}_{3,\lambda})$. $\square$

Proposition 4.4. *The Lie algebra $\mathfrak{r}_4$ has no generalized complex structure.*

Proof. From (17), (21) and (12), we have that with respect to the basis $\{X_i\}$ the only non-zero Lie brackets on $T^*(\mathfrak{r}_4)$ are

$$[X_1, X_2] = X_2, \qquad [X_1, X_3] = X_2 + X_3, \qquad [X_1, X_4] = X_3 + X_4,$$
$$[X_1, X_7] = -X_7 - X_8, \qquad [X_1, X_6] = -X_6 - X_7, \qquad [X_1, X_8] = -X_8,$$
$$[X_2, X_6] = [X_3, X_6] = X_5, \quad [X_3, X_7] = [X_4, X_7] = X_5, \quad [X_4, X_8] = X_5.$$

If there is a hermitian structure $(\mathcal{J}, \langle \cdot , \cdot \rangle)$ on $T^*(\mathfrak{r}_4)$, then

$$0 = N^1_{78} = {a_{18}}^2, \quad 0 = N^1_{67} = {a_{17}}^2 - a_{16} a_{18},$$
$$0 = N^6_{35} = {a_{12}}^2, \quad 0 = N^8_{35} = -{a_{13}}^2 + a_{12} a_{14},$$

imply that $a_{12} = a_{13} = a_{17} = a_{18} = 0$. So, it must be ${a_{16}}^2 + {a_{14}}^2 \neq 0$ since, according to (20), at least an element a_{1j} of the first row of the matrix (19) associated to $\mathcal{J}$ must be non-zero for $j \neq 1$. Now, we consider

$$0 = N^2_{78} = 2a_{16} a_{38}, \qquad 0 = N^8_{78} = 2a_{14} a_{38},$$
$$0 = N^7_{26} = -2a_{16} a_{72}, \qquad 0 = N^7_{24} = -2a_{14} a_{72},$$
$$0 = N^7_{47} = a_{14}(a_{43} - a_{32}), \qquad 0 = N^7_{67} = a_{16}(a_{43} - a_{32}),$$
$$0 = N^3_{15} = a_{14} a_{38} - a_{16} a_{32}, \quad 0 = N^5_{35} = a_{14} a_{43} + a_{16} a_{72}.$$

From these equations, and using that a_{16} and a_{14} do not vanish simultaneously, we conclude that $a_{38} = a_{72} = a_{43} = a_{32} = 0$. Then,

$$0 = N^5_{37} = 1 + {a_{33}}^2 + a_{32}(a_{11} + a_{33} + a_{23}) + a_{43}(a_{33} - a_{11} + a_{34}) - a_{27} a_{72} - a_{83} a_{38},$$

implies that $0 = 1 + {a_{33}}^2$, which is not possible. Thus, $\mathcal{J}$ cannot be integrable. $\square$

Proposition 4.5. *For $\lambda \neq 0, \pm 1$, the Lie algebra $\mathfrak{r}_{4,\lambda}$ does not admit generalized complex structures.*

Proof. With respect to the basis $\{X_i\}$ given by (17), and according to (12) and (21), the only non-zero Lie brackets on $T^*(\mathfrak{r}_{4,\lambda})$ are

$$[X_1, X_2] = X_2, \qquad [X_1, X_3] = \lambda X_3, \qquad [X_1, X_4] = X_3 + \lambda X_4,$$
$$[X_1, X_6] = -X_6, \qquad [X_1, X_7] = -\lambda X_7 - X_8, \qquad [X_1, X_8] = -\lambda X_8,$$
$$[X_2, X_6] = [X_4, X_7] = X_5, \quad [X_3, X_7] = [X_4, X_8] = \lambda X_5.$$

Suppose that, for $\lambda \in \mathbb{R} - \{-1, 0, 1\}$, $T^*(\mathfrak{r}_{4,\lambda})$ has a hermitian structure with complex structure $\mathcal{J}$. Since all the coefficients N_{ij}^k of the Nijenhuis tensor of $\mathcal{J}$ are zero, we have

$$0 = N_{78}^1 = {a_{18}}^2, \quad \text{and} \quad 0 = N_{45}^7 = {a_{13}}^2.$$

Thus, $a_{18} = a_{13} = 0$. Let us consider the equation

$$\begin{aligned} 0 = N_{26}^6 = 1 + {a_{22}}^2 + a_{12}a_{12} + a_{16}a_{61} + a_{23}a_{42} - a_{28}a_{72} \\ + \lambda(a_{23}a_{32} + a_{24}a_{42} - a_{27}a_{72} - a_{28}a_{82}). \end{aligned} \tag{24}$$

Since

$$0 = N_{57}^2 = (\lambda - 1)a_{16}a_{17}, \qquad 0 = N_{67}^4 = 2\lambda a_{16}a_{38} - (1 + \lambda)a_{17}a_{28},$$
$$0 = N_{67}^7 = a_{16}a_{43} - (1 - \lambda)a_{17}a_{23}, \quad 0 = N_{57}^6 = (1 + \lambda)a_{12}a_{17},$$
$$0 = N_{37}^6 = (1 + \lambda)a_{17}a_{72} - a_{12}a_{43}, \quad 0 = N_{78}^6 = 2\lambda a_{12}a_{38} - (1 - \lambda)a_{17}a_{42},$$

we obtain that $a_{17} = 0$ using (24) and the conditions $\lambda \neq 0, 1, -1$. Now, the equations

$$0 = N_{46}^1 = -(1 + \lambda)a_{14}a_{16}, \qquad 0 = N_{46}^4 = (1 + \lambda)a_{14}a_{28} - a_{16}a_{43},$$
$$0 = N_{46}^7 = (1 - \lambda)a_{14}a_{23} + 2\lambda a_{16}a_{83}, \quad 0 = N_{45}^6 = (\lambda - 1)a_{14}a_{12},$$
$$0 = N_{34}^6 = (1 + \lambda)a_{14}a_{72} + 2\lambda a_{12}a_{83}, \quad 0 = N_{25}^5 = a_{12}(a_{11} + a_{22}) + \lambda a_{14}a_{42}$$

and (24) imply that $a_{14} = 0$. Hence ${a_{12}}^2 + {a_{16}}^2 \neq 0$. Morever, we have

$$0 = N_{37}^2 = -a_{16}a_{43}, \qquad 0 = N_{37}^6 = -a_{12}a_{43}, \quad 0 = N_{78}^2 = 2\lambda a_{16}a_{38},$$
$$0 = N_{78}^6 = 2\lambda a_{12}a_{38}, \qquad 0 = N_{56}^6 = 2a_{12}a_{16}, \quad 0 = N_{35}^5 = a_{12}a_{23} + a_{16}a_{72},$$
$$0 = N_{15}^3 = -a_{12}a_{27} - a_{16}a_{32}.$$

Therefore, $a_{43} = a_{38} = 0$ and $a_{23}a_{32} = a_{27}a_{72} = 0$. Taking into account these equalities and

$$0 = N_{37}^5 = \lambda(1 + {a_{33}}^2) + a_{23}a_{32} - a_{27}a_{72} + a_{43}(a_{33} - a_{11} + \lambda a_{34}) - \lambda a_{38}a_{83}$$

we conclude that $1 + a_{33}{}^2 = 0$ because $\lambda \neq 0$. This proves that, for $\lambda \neq -1, 0, 1$, $\mathfrak{r}_{4,\lambda}$ does not admit generalized complex structures. $\square$

Proposition 4.6. *The Lie algebra $\mathfrak{r}_{4,\mu,\lambda}$ has no generalized complex structure for $-1 < \mu < \lambda < 1$ such that $\mu\lambda \neq 0$ and $\mu + \lambda \neq 0$.*

Proof. For $T^*(\mathfrak{r}_{4,\mu,\lambda})$ we take the basis $\{X_i\}$ defined by (17). Then, using (21) and (12) we see that the only non-zero Lie brackets on $T^*(\mathfrak{r}_{4,\mu,\lambda})$ are

$$\begin{aligned} &[X_1, X_2] = X_2, && [X_1, X_3] = \mu X_3, && [X_1, X_4] = \lambda X_4,\\ &[X_2, X_6] = X_5, && [X_3, X_7] = \mu X_5, && [X_4, X_8] = \lambda X_5,\\ &[X_1, X_6] = -X_6, && [X_1, X_7] = -\mu X_7, && [X_1, X_8] = -\lambda X_8. \end{aligned}$$

As in proof of the previous propositions, we assume that $T^*(\mathfrak{r}_{4,\mu,\lambda})$ has a hermitian structure $(\mathcal{J}, \langle \cdot, \cdot \rangle)$. Then,

$$0 = N_{26}^5 = 1 + a_{22}{}^2 + a_{21}a_{12} + a_{16}a_{61} + \mu(a_{23}a_{32} - a_{27}a_{72}) + \lambda(a_{24}a_{42} - a_{28}a_{82}). \tag{25}$$

Consider

$$\begin{aligned} &0 = N_{68}^1 = (1 - \lambda)a_{16}a_{18}, \quad 0 = N_{58}^6 = (1 + \lambda)a_{12}a_{18},\\ &0 = N_{58}^3 = (\lambda - \mu)a_{17}a_{18}, \quad 0 = N_{58}^8 = 2\lambda a_{14}a_{18},\\ &0 = N_{48}^6 = (\lambda - 1)a_{14}a_{42} + (1 + \lambda)a_{18}a_{82}, \quad 0 = N_{48}^2 = (1 - \lambda)a_{18}a_{24} - (1 + \lambda)a_{14}a_{28},\\ &0 = N_{78}^2 = (1 + \mu)a_{18}a_{27} + (\lambda + \mu)a_{16}a_{38} - (1 + \lambda)a_{17}a_{28},\\ &0 = N_{78}^6 = (1 - \mu)a_{18}a_{32} + (\lambda + \mu)a_{12}a_{38} + (\lambda - 1)a_{17}a_{42}. \end{aligned}$$

From these equations and (25), we obtain $a_{18} = 0$. Now, we have

$$\begin{aligned} &0 = N_{67}^1 = (1 - \mu)a_{16}a_{17}, \quad 0 = N_{57}^6 = (1 + \mu)a_{12}a_{17},\\ &0 = N_{68}^3 = (1 + \lambda)a_{17}a_{28} - (\lambda + \mu)a_{16}a_{38}, \quad 0 = N_{78}^6 = (\lambda + \mu)a_{12}a_{38} + (\lambda - 1)a_{17}a_{42},\\ &0 = N_{37}^1 = -2\mu a_{17}a_{13}, \quad 0 = N_{37}^6 = (\mu - 1)a_{13}a_{32} + (1 + \mu)a_{17}a_{72},\\ &0 = N_{37}^2 = (1 - \mu)a_{17}a_{23} - (1 + \mu)a_{13}a_{27}. \end{aligned}$$

Then, using (25), we see that $a_{17} = 0$. From the equations

$$\begin{aligned} &0 = N_{68}^8 = (1 + \lambda)a_{14}a_{28}, \quad 0 = N_{48}^6 = (\lambda - 1)a_{14}a_{42},\\ &0 = N_{56}^8 = (1 + \lambda)a_{14}a_{16}, \quad 0 = N_{47}^2 = -(1 + \mu)a_{14}a_{27} + (\lambda - \mu)a_{16}a_{34},\\ &0 = N_{25}^8 = (\lambda - 1)a_{12}a_{14}, \quad 0 = N_{27}^8 = (1 - \mu)a_{14}a_{32} + (\mu - \lambda)a_{12}a_{34}, \end{aligned}$$

and (25), we conclude that $a_{14} = 0$. Moreover, we have

$$0 = N^2_{37} = -(1+\mu)a_{13}a_{27}, \quad 0 = N^6_{37} = (\mu - 1)a_{13}a_{32},$$
$$0 = N^1_{36} = -(1+\mu)a_{13}a_{16}, \quad 0 = N^4_{36} = (1+\lambda)a_{13}a_{28} + (\lambda - \mu)a_{16}a_{43},$$
$$0 = N^6_{35} = (\mu - 1)a_{12}a_{13}, \quad 0 = N^8_{36} = (1-\lambda)a_{13}a_{24} - (\lambda + \mu)a_{16}a_{83},$$

which imply that $a_{13} = 0$ using again (25). Thus, ${a_{12}}^2 + {a_{16}}^2 \neq 0$. Now, taking account the equations

$$0 = N^6_{56} = 2a_{12}a_{16},$$
$$0 = N^2_{78} = (\lambda + \mu)a_{16}a_{38}, \quad 0 = N^6_{78} = (\lambda + \mu)a_{12}a_{38},$$
$$0 = N^2_{38} = (\mu - \lambda)a_{16}a_{43}, \quad 0 = N^6_{38} = (\mu - \lambda)a_{12}a_{43},$$
$$0 = N^6_{15} = -a_{12}a_{24} - a_{16}a_{82}, \quad 0 = N^4_{15} = -a_{12}a_{28} - a_{16}a_{42},$$

we have that $a_{38} = a_{43} = 0$ and $a_{24}a_{42} = a_{28}a_{82} = 0$. So,

$$0 = N^5_{48} = \lambda(1 + {a_{44}}^2) + \mu(a_{34}a_{43} - a_{38}a_{83}) + a_{24}a_{42} - a_{28}a_{82} = \lambda(1 + {a_{44}}^2).$$

This implies that $\lambda = 0$ or $1 + {a_{44}}^2 = 0$, which is not possible. This completes the proof. □

Let $\mathfrak{g}$ be an arbitrary Lie algebra. Denote by $b_i(\mathfrak{g})$ the dimension of the i-th cohomology group $H^i(\mathfrak{g})$ of $\mathfrak{g}$, by $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ the derived subalgebra and by $\mathfrak{z}(\mathfrak{g})$ the center of $\mathfrak{g}$. We recall that $\mathfrak{g}$ is called *completely solvable* when $\mathfrak{g}$ is solvable and $\mathrm{ad}(x)$ has only real eigenvalues for any $x \in \mathfrak{g}$.

Theorem 4.7. *Let G be a four dimensional solvable Lie group with Lie algebra $\mathfrak{g}$. Then the following statements are equivalent*:

(i) *G has no left invariant generalized complex structure*;

(ii) *G admits neither left invariant symplectic nor left invariant complex structures*;

(iii) *$\mathfrak{g}$ is completely solvable and one of the two following conditions is satisfied*:

(*a*) $b_1(T^*\mathfrak{g}) = 3$, $b_3(T^*\mathfrak{g}) = 5$, $b_1(\mathfrak{g}/\mathfrak{z}(\mathfrak{g}')) = 1$, *or*

(*b*) $b_1(T^*\mathfrak{g}) = 1$, $b_3(T^*\mathfrak{g}) = 2$.

Proof. Clearly (i) implies (ii). The converse follows from Propositions 4.2 to 4.6. The calculation of the numbers $b_i(T^*(\mathfrak{g}))$, $(i = 1, 3)$, and $b_1(\mathfrak{g}/\mathfrak{z}(\mathfrak{g}'))$, where $\mathfrak{g}$ is a four dimensional solvable Lie algebra, shows that condition (iii) is satisfied if and only if (i) holds (see Table in Section 5.2). □

5. Generalized complex structures of type 1 on solvable Lie groups

In this section we exhibit the four dimensional solvable Lie algebras which have a generalized complex structure of type 1; Theorem 4.7 implies that they admit either symplectic or complex structures. We also study necessary and sufficient conditions on a four dimensional solvable Lie algebra $\mathfrak{g}$ to admit generalized complex structures of type 1. As a consequence of our results, we obtain in Corollary 5.6 a condition, involving the odd numbers $b_i(T^*(\mathfrak{g}))$, for the non-existence of such structures.

5.1. Existence. First, we list below the family of Lie algebras having either symplectic or complex structures. Such Lie algebras together with those shown in (21), exhaust the class of four dimensional solvable Lie algebras (see [2, 9, 16, 19]).

(26)

$$
\begin{aligned}
\mathfrak{aff}(\mathbb{R})\times\mathfrak{aff}(\mathbb{R})&: [e_0,e_1]=e_1,\ [e_2,e_3]=e_3;\\
\mathfrak{aff}(\mathbb{C})&: [e_0,e_2]=e_2,\ [e_0,e_3]=e_3,\ [e_1,e_2]=e_3,\ [e_1,e_3]=-e_2;\\
\mathbb{R}\times\mathfrak{e}(2)&: [e_1,e_2]=-e_3,\ [e_1,e_3]=e_2;\\
\mathbb{R}\times\mathfrak{h}_3&: [e_1,e_2]=e_3;\\
\mathbb{R}\times\mathfrak{r}_{3,\lambda}&: [e_1,e_2]=e_2,\ [e_1,e_3]=\lambda e_3,\quad \lambda\in\{-1,0,1\};\\
\mathfrak{r}_{4,\lambda}&: [e_0,e_1]=e_1,\ [e_0,e_2]=\lambda e_2,\ [e_0,e_3]=e_2+\lambda e_3,\ \lambda\in\{-1,0,1\};\\
\mathfrak{r}_{4,\mu,1}&: [e_0,e_1]=e_1,\ [e_0,e_2]=\mu e_2,\ [e_0,e_3]=e_3,\quad -1<\mu\leq 1,\ \mu\neq 0;\\
\mathfrak{r}_{4,\mu,\mu}&: [e_0,e_1]=e_1,\ [e_0,e_2]=\mu e_2,\ [e_0,e_3]=\mu e_3,\quad -1<\mu<1,\ \mu\neq 0;\\
\mathfrak{r}_{4,\mu,-\mu}&: [e_0,e_1]=e_1,\ [e_0,e_2]=\mu e_2,\ [e_0,e_3]=-\mu e_3,\quad -1<\mu<0;\\
\mathfrak{r}_{4,-1,\lambda}&: [e_0,e_1]=e_1,\ [e_0,e_2]=-e_2,\ [e_0,e_3]=\lambda e_3,\quad -1<\lambda<0;\\
\mathfrak{r}_{4,-1,-1}&: [e_0,e_1]=e_1,\ [e_0,e_2]=-e_2,\ [e_0,e_3]=-e_3;\\
\mathbb{R}\times\mathfrak{r}'_{3,\lambda}&: [e_1,e_2]=\lambda e_2-e_3,\ [e_1,e_3]=e_2+\lambda e_3,\quad \lambda>0;\\
\mathfrak{n}_4&: [e_0,e_1]=e_2,\ [e_0,e_2]=e_3;\\
\mathfrak{r}'_{4,\mu,\lambda}&: [e_0,e_1]=\mu e_1,\ [e_0,e_2]=\lambda e_2-e_3,\ [e_0,e_3]=e_2+\lambda e_3,\quad \mu>0,\ \lambda\in\mathbb{R};\\
\mathfrak{d}_4&: [e_0,e_1]=e_1,\ [e_0,e_2]=-e_2,\ [e_1,e_2]=e_3;\\
\mathfrak{d}_{4,\lambda}&: [e_0,e_1]=\lambda e_1,\ [e_0,e_2]=(1-\lambda)e_2,\ [e_0,e_3]=e_3,\ [e_1,e_2]=e_3,\quad \lambda\geq\frac{1}{2};\\
\mathfrak{d}'_{4,\lambda}&: [e_0,e_1]=\lambda e_1-e_2,\ [e_0,e_2]=e_1+\lambda e_2,\ [e_0,e_3]=2\lambda e_3,\ [e_1,e_2]=e_3,\\
&\quad\ \lambda\geq 0;\\
\mathfrak{h}_4&: [e_0,e_1]=e_1,\ [e_0,e_2]=e_1+e_2,\ [e_0,e_3]=2e_3,\ [e_1,e_2]=e_3.
\end{aligned}
$$

Proposition 5.1. *The Lie algebras* $\mathfrak{aff}(\mathbb{R})\times\mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{C})$, $\mathbb{R}\times\mathfrak{e}(2)$, $\mathbb{R}\times\mathfrak{h}_3$, $\mathbb{R}\times\mathfrak{r}_{3,0}$, $\mathfrak{r}_{4,-1,-1}$, $\mathfrak{r}'_{4,\mu,0}$ $(\mu>0)$, $\mathfrak{d}_{4,1/2}$, $\mathfrak{d}_{4,2}$ *and* $\mathfrak{d}'_{4,\lambda}$ $(\lambda>0)$ *admit generalized complex structures of type* 0, 1 *and* 2.

Proof. It follows from results in [15, 18, 23] that all of the above Lie algebras admit both symplectic and complex structures, which give rise to generalized complex structures of type 0 and 2, respectively. A generalized complex structure of type 1 on $\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$ can be obtained by combining one of type 0 with one of type 1 on $\mathfrak{aff}(\mathbb{R})$ (see (15) and (16)). For the remaining Lie algebras, we exhibit a generalized complex structure of type 1.

$$\begin{aligned} \mathfrak{aff}(\mathbb{C}),\ \mathfrak{r}_{4,-1,-1}&: \mathcal{J}(e_0) = \alpha^1,\ \mathcal{J}(e_2) = e_3 \\ \mathbb{R} \times \mathfrak{e}(2),\ \mathbb{R} \times \mathfrak{r}_{3,0},\ \mathfrak{d}_{4,1/2},\ \mathfrak{d}'_{4,\lambda}\ (\lambda > 0)&: \mathcal{J}(e_0) = \alpha^3,\ \mathcal{J}(e_1) = e_2, \\ \mathbb{R} \times \mathfrak{h}_3,\ \mathfrak{r}'_{4,\mu,0}\ (\mu > 0),\ \mathfrak{d}_{4,2}&: \mathcal{J}(e_0) = e_1,\ \mathcal{J}(e_2) = \alpha^3. \end{aligned} \tag{27}$$ $\square$

Proposition 5.2. *The Lie algebras* $\mathbb{R} \times \mathfrak{r}_{3,-1}$, $\mathfrak{r}_{4,-1}$, $\mathfrak{r}_{4,0}$, $\mathfrak{n}_4$, $\mathfrak{r}_{4,\mu,-\mu}$ $(-1 < \mu < 0)$ *and* $\mathfrak{r}_{4,-1,\lambda}$ $(-1 \leq \lambda < 0)$ *admit generalized complex structures of type* 0 *and* 1*, but not of type* 2.

Proof. First we notice that every Lie algebra mentioned in the proposition has symplectic structures but does not admit complex structures ([15, 17, 18, 23]), so it does not possess generalized complex structures of type 2 (Theorem 4.1). For each one of these Lie algebras, we show a generalized complex structure of type 1:

$$\begin{aligned} \mathbb{R} \times \mathfrak{r}_{3,-1},\ \mathfrak{r}_{4,0},\ \mathfrak{r}_{4,\mu,-\mu},\ \mathfrak{n}_4&: \mathcal{J}(e_0) = e_1,\ \mathcal{J}(e_2) = \alpha^3, \\ \mathfrak{r}_{4,-1},\ \mathfrak{r}_{4,-1,\lambda}&: \mathcal{J}(e_0) = e_3,\ \ \mathcal{J}(e_1) = \alpha^2. \end{aligned} \tag{28}$$ $\square$

Proposition 5.3. *The Lie algebras* $\mathbb{R} \times \mathfrak{r}_{3,1}$, $\mathfrak{r}_{4,1}$, $\mathfrak{r}'_{4,\mu,\lambda}$ $(\mu > 0,\ \lambda \neq 0)$, $\mathbb{R} \times \mathfrak{r}'_{3,\lambda}$ $(\lambda \neq 0)$, $\mathfrak{r}_{4,\mu,\mu}$ $(-1 < \mu \leq 1,\ \mu \neq 0)$, $\mathfrak{r}_{4,\mu,1}$ $(-1 < \mu \leq 1,\ \mu \neq 0)$*, and* $\mathfrak{d}'_{4,0}$ *admit generalized complex structures of type* 1 *and* 2*, but not of type* 0.

Proof. These Lie algebras have complex structures and do not admit symplectic structures ([15, 17, 18, 23]), thus they admit generalized complex structures of type 2 but not of type 0. A generalized complex structure of type 1 is given by

$$\begin{aligned} \mathbb{R} \times \mathfrak{r}_{3,1},\ \mathfrak{r}_{4,\mu,\mu},\ \mathfrak{r}'_{4,\mu,\lambda}&: \mathcal{J}(e_0) = \alpha^1,\ \mathcal{J}(e_2) = e_3, \\ \mathfrak{r}_{4,1}, \mathfrak{r}_{4,\lambda,1}&: \mathcal{J}(e_0) = \alpha^2,\ \mathcal{J}(e_1) = e_3, \\ \mathbb{R} \times \mathfrak{r}'_{3,\lambda},\ \mathfrak{d}'_{4,0}&: \mathcal{J}(e_0) = \alpha^3,\ \mathcal{J}(e_1) = e_2, \end{aligned} \tag{29}$$ $\square$

5.2. Obstructions.

Proposition 5.4. $\mathfrak{d}_4$ *is the unique four dimensional solvable Lie algebra admitting generalized complex structures of type* 2*, but not of type* 0 *or* 1.

Proof. We consider the basis $\{X_i\}$ defined by (17) for $T^*(\mathfrak{d}_4)$. Taking into account (12) and the structure equations of the Lie algebra $\mathfrak{d}_4$ given in (26), we see that the only non-zero brackets on $T^*(\mathfrak{d}_4)$ are

$$\begin{aligned} &[X_1, X_2] = X_2, && [X_1, X_3] = -X_3, \\ &[X_2, X_3] = X_4, && [X_2, X_6] = -[X_3, X_7] = X_5, \\ &-[X_1, X_6] = [X_3, X_7] = X_6, && [X_1, X_7] = -[X_2, X_8] = X_7. \end{aligned}$$

Suppose that $\mathcal{J}$ is a generalized complex structure on $\mathfrak{d}_4$. Let us consider the equations

$$\begin{aligned} &0 = N^7_{56} = a_{16}a_{24}, \quad 0 = N^8_{56} = a_{16}a_{14}, \quad 0 = N^3_{46} = -a_{34}a_{16} - a_{24}a_{17}, \\ &0 = N^8_{26} = a_{24}(a_{12} - a_{34}) - a_{16}a_{82}, \\ &0 = N^8_{36} = -2a_{14}a_{23} - a_{13}a_{24} + {a_{24}}^2 + a_{16}a_{83}. \end{aligned}$$

The condition $\mathcal{J}^2 = -\operatorname{Id}$ implies that

$$\begin{aligned} 0 &= (\mathcal{J}^2)^4_2 = a_{14}a_{21} + a_{24}(a_{22} + a_{44}) + a_{23}a_{34} + a_{16}a_{81} - a_{27}a_{83}, \\ -1 &= (\mathcal{J}^2)^4_4 = a_{14}a_{41} + a_{24}a_{42} + a_{34}a_{43} + {a_{44}}^2 + a_{18}a_{81} + a_{28}a_{82} + a_{38}a_{83}, \end{aligned}$$

and so we obtain $a_{16} = 0$. Now, from the equations

$$\begin{aligned} &0 = N^1_{47} = a_{14}a_{17}, \quad 0 = N^2_{47} = a_{17}a_{24}, \quad 0 = N^6_{57} = -a_{34}a_{17}, \\ &0 = N^7_{47} = a_{34}(a_{13} - a_{24}) - a_{17}a_{83}, \\ &0 = N^6_{47} = -2a_{14}a_{32} - a_{12}a_{34} + {a_{34}}^2 + a_{17}a_{82}, \end{aligned}$$

and, from $\mathcal{J}^2 = -\operatorname{Id}$,

$$\begin{aligned} 0 &= (\mathcal{J}^2)^4_2 = a_{14}a_{31} + a_{24}a_{32} + a_{34}(a_{33} + a_{44}) + a_{17}a_{81} + a_{27}a_{82}, \\ -1 &= (\mathcal{J}^2)^4_4 = a_{14}a_{41} + a_{24}a_{42} + a_{34}a_{43} + {a_{44}}^2 + a_{18}a_{81} + a_{28}a_{82} + a_{38}a_{83}, \end{aligned}$$

we obtain $a_{17} = 0$. Moreover, $a_{27} = 0$ because $N^3_{78} = -{a_{27}}^2 = 0$.

The equations

$$\begin{aligned} &0 = (\mathcal{J}^2)^5_1 = -2a_{14}a_{18}, \quad 0 = N^8_{78} = a_{18}a_{34} - a_{14}a_{38}, \quad 0 = N^7_{58} = a_{14}a_{28} - a_{13}a_{18}, \\ &0 = N^8_{68} = a_{14}a_{28} - a_{18}a_{24}, \quad 0 = N^7_{68} = a_{28}(a_{13} + a_{24}) - a_{18}a_{23}, \\ &0 = N^7_{17} = -1 + a_{13}a_{31} - a_{23}a_{32} + {a_{33}}^2 - a_{21}a_{34}, \end{aligned}$$

imply that $a_{18} = 0$. But, since

$$0 = (\mathcal{J}^2)^6_2 = -2a_{24}a_{28}, \quad 0 = N^7_{68} = a_{28}(a_{13} + a_{24}),$$
$$0 = (\mathcal{J}^2)^6_1 = -a_{14}a_{28}, \quad 0 = (\mathcal{J}^2)^8_1 = a_{12}a_{28} + a_{13}a_{38},$$
$$-1 = (\mathcal{J}^2)^1_1 = {a_{11}}^2 + a_{12}a_{21} + a_{13}a_{31} + a_{14}a_{41},$$

we obtain $a_{28} = 0$. And finally, from the equations

$$0 = (\mathcal{J}^2)^8_1 = a_{23}a_{38}, \quad 0 = N^2_{18} = -a_{38}a_{13}, \quad 0 = (\mathcal{J}^2)^7_3 = -2a_{34}a_{38},$$
$$0 = N^7_{17} = 1 + {a_{33}}^2 + a_{13}a_{31} - a_{23}a_{32} - a_{21}a_{34},$$

we have $a_{38} = 0$. So the matrix $\mathcal{J}_2$ in (18) is the null matrix, and then $\mathfrak{d}_4$ does not admit generalized complex structures of types 0 and 1. The almost complex structure defined by $J(e_0) = e_1$ and $J(e_3) = e_2$ is integrable and thus $\mathfrak{d}_4$ admits a generalized complex structure of type 2. The uniqueness is seen in the table at the end of this section. □

Proposition 5.5. *The Lie algebras* $\mathfrak{d}_{4,\lambda}$, $(\lambda \neq 1/2, 2)$ *and* $\mathfrak{h}_4$ *admit generalized complex structures of type* 0 *and* 2, *but not of type* 1.

Proof. Doing a similar calculation to that made in the previous proposition, one can check that for a generalized complex structure on $\mathfrak{h}_4$, the matrices $\mathcal{J}_1$ and $\mathcal{J}_2$ in (18) are

$$\mathcal{J}_1 = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & a_{13} \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & 0 & 0 & a_{18} \\ 0 & 0 & 2a_{18} & a_{28} \\ 0 & -2a_{18} & 0 & a_{38} \\ -a_{18} & -a_{28} & -a_{38} & 0 \end{pmatrix}.$$

From $\mathcal{J}^2 = -\operatorname{Id}$ we have ${a_{18}}^2 + a^2_{13} \neq 0$. Since $0 = N^8_{68} = 3a_{18}a_{13}$, we have the two following possibilities:

- $a_{18} \neq 0$, $a_{13} = 0$. Then, $\operatorname{rank}\mathcal{J}_2 = 4$ and the possible generalized complex structures are of type 0; for example,

$$\mathcal{J}(e_0) = 2\alpha^3, \quad \mathcal{J}(e_1) = \alpha^2.$$

- $a_{13} \neq 0$, $a_{18} = 0$. In this case, we obtain $a_{38} = a_{28} = 0$ using $0 = N^7_{78} = 3a_{13}a_{38}$ and $0 = N^7_{68} = a_{13}(4a_{28} + a_{38})$. So $\mathcal{J}_2 \equiv 0$ and the possible generalized complex structures are of type 2; for example,

$$\mathcal{J}(e_0) = e_2, \quad \mathcal{J}(e_1) = e_3.$$

For a generalized complex structure on the Lie algebra $\mathfrak{d}_{4,\lambda}$, $(\lambda \neq 1/2, 1, 2)$, the matrices $\mathcal{J}_1$ and $\mathcal{J}_2$, given by (18), are

$$\mathcal{J}_1 = \begin{pmatrix} a_{11} & a_{12} & a_{1,3} & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & 0 & 0 & a_{18} \\ 0 & 0 & a_{18} & a_{28} \\ 0 & -a_{18} & 0 & a_{38} \\ -a_{18} & -a_{28} & -a_{38} & 0 \end{pmatrix}.$$

We consider two possibilities according to $a_{18} \neq 0$ or $a_{18} = 0$:

- If $a_{18} \neq 0$, rank $\mathcal{J}_2 = 4$ and the generalized complex structures are of type 0; for example,

$$\mathcal{J}(e_0) = \alpha^3, \quad \mathcal{J}(e_1) = \alpha^2.$$

- If $a_{18} = 0$, then ${a_{12}}^2 + {a_{13}}^2 \neq 0$. Since $0 = N_{35}^6 = (1 - 2\lambda)a_{12}a_{13}$, we consider two subcases

A) $a_{12} \neq 0$, $a_{13} = 0$. From $N_{58}^5 = -\lambda a_{12}a_{28}$ we obtain $a_{28} = 0$ and from $0 = N_{78}^6 = -a_{38}(a_{34} - a_{12}(\lambda - 2))$ and $0 = N_{47}^6 = a_{34}(\lambda a_{12} + a_{34})$ we obtain $a_{38} = 0$. Hence $\mathcal{J}_2 \equiv 0$ and the generalized complex structures are of type 2; for example,

$$\mathcal{J}(e_0) = \lambda e_1, \quad \mathcal{J}(e_2) = -e_3.$$

B) $a_{13} \neq 0$, $a_{12} = 0$. From $N_{58}^5 = (-1 + \lambda)a_{13}a_{38}$ we obtain $a_{38} = 0$ and from $0 = N_{68}^7 = a_{28}(a_{24} + a_{13}(\lambda + 1))$ and $0 = N_{46}^7 = -a_{24}(a_{24} + a_{13}(\lambda - 1))$ we obtain $a_{28} = 0$. So, $\mathcal{J}_2 \equiv 0$ and the generalized complex structures are of type 2; for example,

$$\mathcal{J}(e_0) = (1 - \lambda)e_2, \quad \mathcal{J}(e_1) = e_3.$$

For a generalized complex structure on the Lie algebra $\mathfrak{d}_{4,1}$, the matrices $\mathcal{J}_1$ and $\mathcal{J}_2$ in (18) are

$$\mathcal{J}_1 = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & -a_{12} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & 0 & 0 & a_{18} \\ 0 & 0 & a_{18} & a_{28} \\ 0 & -a_{18} & 0 & a_{38} \\ -a_{18} & -a_{28} & -a_{38} & 0 \end{pmatrix}.$$

Therefore, ${a_{12}}^2 + {a_{18}}^2 \neq 0$. Since $0 = N_{58}^6 = 2a_{12}a_{18}$ we consider the two following possibilities:

- $a_{18} \neq 0$, $a_{12} = 0$. Then, rank $\mathcal{J}_2 = 4$ and the generalized complex structures are of type 0; for example,

$$\mathcal{J}(e_0) = \alpha^3, \quad \mathcal{J}(e_1) = \alpha^2.$$

• $a_{18} = 0$, $a_{12} \neq 0$. Because $N^7_{78} = -a_{12}a_{28}$ and $0 = N^6_{78} = 2a_{38}a_{12}$, we have $a_{28} = a_{38} = 0$. So, $\mathcal{J}_2 \equiv 0$ and the generalized complex structures are of type 2; for example,

$$\mathcal{J}(e_0) = e_1, \quad \mathcal{J}(e_2) = -e_3. \qquad \square$$

The previous propositions together with Table 1 imply the next result.

Corollary 5.6. *Let $\mathfrak{g}$ be a four dimensional Lie algebra admitting a generalized complex structure. Then, $\mathfrak{g}$ does not admit a generalized complex structure of type* 1 *if and only if $\mathfrak{g}$ is completely solvable and one of the following conditions is satisfied*:
(i) $b_1(T^*\mathfrak{g}) = b_3(T^*\mathfrak{g}) = 1$, *or*
(ii) $b_1(T^*\mathfrak{g}) = 2$, $b_3(T^*\mathfrak{g}) = 4$.

REMARK 2. We must notice that the Lie algebra $\mathfrak{d}'_{4,\lambda}$ satisfies $b_1(T^*\mathfrak{d}'_{4,\lambda}) = b_3(T^*\mathfrak{d}'_{4,\lambda}) = 1$, but it is not completely solvable. Therefore, according to the previous Corollary, it has generalized complex structures of type 1. In general, in the table below, the Lie algebras $\mathfrak{g}'$ are not completely solvable, so they have generalized complex structures of type 1.

In the table below we summarize the previous results and, for each solvable Lie algebra admitting generalized complex structures, we exhibit *one* of the simplest examples of each type (—stands for non existence).

Table 1.

$\mathfrak{g}$	$b_1(T^*\mathfrak{g})$	$b_3(T^*\mathfrak{g})$	Type 0	Type 1	Type 2
$\mathbb{R} \times \mathfrak{r}_3$	3	5	—	—	—
$\mathbb{R} \times \mathfrak{r}_{3,\lambda}$ $\lvert\lambda\rvert < 1,\ \lambda \neq 0$	3	5	—	—	—
$\mathfrak{r}_4$	1	2	—	—	—
$\mathfrak{r}_{4,\lambda}$ $\lambda \in \mathbb{R},\ \lambda \neq -1, 0, 1$	1	2	—	—	—
$\mathfrak{r}_{4,\mu,\lambda}$ $-1 < \mu < \lambda < 1$ $\mu\lambda \neq 0,\ \mu + \lambda \neq 0$	1	2	—	—	—
$\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$	3	5	$\mathcal{J}(e_0) = \alpha^1$, $\mathcal{J}(e_2) = \alpha^3$	$\mathcal{J}(e_0) = \alpha^1$, $\mathcal{J}(e_2) = e_3$	$\mathcal{J}(e_0) = e_1$, $\mathcal{J}(e_2) = e_3$
$\mathfrak{aff}(\mathbb{C})$	2	2	$\mathcal{J}(e_0) = \alpha^3$, $\mathcal{J}(e_1) = \alpha^2$	$\mathcal{J}(e_0) = \alpha^1$, $\mathcal{J}(e_2) = e_3$	$\mathcal{J}(e_0) = e_3$, $\mathcal{J}(e_1) = -e_2$
$\mathbb{R} \times \mathfrak{h}_3$	5	31	$\mathcal{J}(e_0) = \alpha^2$, $\mathcal{J}(e_1) = \alpha^3$	$\mathcal{J}(e_0) = e_1$, $\mathcal{J}(e_2) = \alpha^3$	$\mathcal{J}(e_0) = e_1$, $\mathcal{J}(e_2) = e_3$

$\mathfrak{g}$	$b_1(T^*\mathfrak{g})$	$b_3(T^*\mathfrak{g})$	Type 0	Type 1	Type 2
$\mathbb{R}\times\mathfrak{r}_{3,-1}$	3	13	$\mathcal{J}(e_0)=\alpha^2$, $\mathcal{J}(e_1)=-\alpha^3$	$\mathcal{J}(e_0)=e_1$, $\mathcal{J}(e_2)=\alpha^3$	—
$\mathbb{R}\times\mathfrak{r}_{3,0}$	5	11	$\mathcal{J}(e_0)=\alpha^2$, $\mathcal{J}(e_1)=-\alpha^3$	$\mathcal{J}(e_0)=\alpha^3$, $\mathcal{J}(e_1)=e_2$	$\mathcal{J}(e_0)=e_3$, $\mathcal{J}(e_1)=e_2$
$\mathbb{R}\times\mathfrak{r}_{3,1}$	3	13	—	$\mathcal{J}(e_0)=\alpha^1$, $\mathcal{J}(e_2)=e_3$	$\mathcal{J}(e_0)=e_1$, $\mathcal{J}(e_2)=e_3$
$\mathfrak{r}_{4,-1}$	1	4	$\mathcal{J}(e_0)=\alpha^2$, $\mathcal{J}(e_1)=\alpha^3$	$\mathcal{J}(e_0)=e_3$, $\mathcal{J}(e_1)=\alpha^2$	—
$\mathfrak{r}_{4,0}$	3	7	$\mathcal{J}(e_0)=\alpha^1$, $\mathcal{J}(e_2)=\alpha^3$	$\mathcal{J}(e_0)=e_1$, $\mathcal{J}(e_2)=\alpha^3$	—
$\mathfrak{r}_{4,1}$	1	4	—	$\mathcal{J}(e_0)=\alpha^2$, $\mathcal{J}(e_1)=e_3$	$\mathcal{J}(e_0)=e_3$, $\mathcal{J}(e_1)=e_2$
$\mathfrak{r}_{4,\mu,1}$, $-1<\mu\le 1,\ \mu\neq 0$	1	4	—	$\mathcal{J}(e_0)=\alpha^2$, $\mathcal{J}(e_1)=e_3$	$\mathcal{J}(e_0)=e_2$, $\mathcal{J}(e_1)=e_3$
$\mathfrak{r}_{4,\mu,\mu}$, $-1<\mu\le 1,\ \mu\neq 0$	1	4	—	$\mathcal{J}(e_0)=\alpha^1$, $\mathcal{J}(e_2)=e_3$	$\mathcal{J}(e_0)=e_1$, $\mathcal{J}(e_2)=e_3$
$\mathfrak{r}_{4,-1,\lambda}$, $-1<\lambda<0$	1	4	$\mathcal{J}(e_0)=\alpha^3$, $\mathcal{J}(e_1)=\alpha^2$	$\mathcal{J}(e_0)=e_3$, $\mathcal{J}(e_1)=\alpha^2$	—
$\mathfrak{r}_{4,-1,-1}$,	1	4	$\mathcal{J}(e_0)=\alpha^3$, $\mathcal{J}(e_1)=\alpha^2$	$\mathcal{J}(e_0)=\alpha^1$, $\mathcal{J}(e_2)=e_3$	$\mathcal{J}(e_0)=e_1$, $\mathcal{J}(e_2)=e_3$
$\mathbb{R}\times\mathfrak{r}'_{3,0}$	5	11	$\mathcal{J}(e_0)=\alpha^3$, $\mathcal{J}(e_1)=\alpha^2$	$\mathcal{J}(e_0)=\alpha^3$, $\mathcal{J}(e_1)=e_2$	$\mathcal{J}(e_0)=e_3$, $\mathcal{J}(e_1)=e_2$
$\mathbb{R}\times\mathfrak{r}'_{3,\lambda}$, $\lambda>0$	3	5	—	$\mathcal{J}(e_0)=\alpha^3$, $\mathcal{J}(e_1)=e_2$	$\mathcal{J}(e_0)=e_3$, $\mathcal{J}(e_1)=e_2$

$\mathfrak{g}$	$b_1(T^*\mathfrak{g})$	$b_3(T^*\mathfrak{g})$	Type 0	Type 1	Type 2
$\mathfrak{n}_4$	3	14	$\mathcal{J}(e_0) = \alpha^3$, $\mathcal{J}(e_1) = \alpha^2$	$\mathcal{J}(e_0) = e_1$, $\mathcal{J}(e_2) = \alpha^3$	—
$\mathfrak{r}'_{4,\mu,0}$, $\mu > 0$	1	4	$\mathcal{J}(e_0) = \alpha^1$, $\mathcal{J}(e_2) = \alpha^3$	$\mathcal{J}(e_0) = e_1$, $\mathcal{J}(e_2) = \alpha^3$	$\mathcal{J}(e_0) = e_1$, $\mathcal{J}(e_2) = e_3$
$\mathfrak{r}'_{4,\mu,\lambda}$, $\mu > 0,\ \lambda \neq 0$	1	2	—	$\mathcal{J}(e_0) = \alpha^1$, $\mathcal{J}(e_2) = e_3$	$\mathcal{J}(e_0) = e_1$, $\mathcal{J}(e_2) = e_3$
$\mathfrak{h}_4$	1	1	$\mathcal{J}(e_0) = 2\alpha^3$, $\mathcal{J}(e_1) = \alpha^2$	—	$\mathcal{J}(e_0) = e_2$, $\mathcal{J}(e_1) = e_3$
$\mathfrak{d}_4$	2	4	—	—	$\mathcal{J}(e_0) = e_1$, $\mathcal{J}(e_3) = e_2$
$\mathfrak{d}_{4,\lambda}$ $\lambda > \frac{1}{2},\ \lambda \neq 1, 2$	1	1	$\mathcal{J}(e_0) = \alpha^3$, $\mathcal{J}(e_1) = \alpha^2$	—	$\mathcal{J}(e_0) = \lambda e_1$, $\mathcal{J}(e_2) = -e_3$
$\mathfrak{d}_{4,1}$	2	4	$\mathcal{J}(e_0) = \alpha^3$, $\mathcal{J}(e_1) = \alpha^2$	—	$\mathcal{J}(e_0) = e_1$, $\mathcal{J}(e_2) = -e_3$
$\mathfrak{d}_{4,1/2}$	1	3	$\mathcal{J}(e_0) = \alpha^3$, $\mathcal{J}(e_1) = \alpha^2$	$\mathcal{J}(e_0) = \alpha^3$, $\mathcal{J}(e_1) = e_2$	$\mathcal{J}(e_0) = \frac{1}{2}e_2$, $\mathcal{J}(e_1) = e_3$
$\mathfrak{d}_{4,2}$	1	3	$\mathcal{J}(e_0) = \alpha^3$, $\mathcal{J}(e_1) = \alpha^2$	$\mathcal{J}(e_0) = e_1$, $\mathcal{J}(e_2) = \alpha^3$	$\mathcal{J}(e_0) = e_1$, $\mathcal{J}(e_2) = -\frac{1}{2}e_3$
$\mathfrak{d}'_{4,0}$	2	4	—	$\mathcal{J}(e_0) = \alpha^3$, $\mathcal{J}(e_1) = e_2$	$\mathcal{J}(e_0) = e_3$, $\mathcal{J}(e_1) = e_2$
$\mathfrak{d}'_{4,\lambda}$ $\lambda > 0$	1	1	$\mathcal{J}(e_0) = 2\lambda\alpha^3$, $\mathcal{J}(e_1) = \alpha^2$	$\mathcal{J}(e_0) = \alpha^3$, $\mathcal{J}(e_1) = e_2$	$\mathcal{J}(e_0) = e_3$, $\mathcal{J}(e_1) = e_2$

6. An example in dimension 6

In this section we exhibit an example of a six dimensional (non-nilpotent) solvable Lie algebra $\mathfrak{g}_6$ admitting neither symplectic nor complex structures but having generalized complex structures of types 1 and 2. This proves that Theorem 4.1 fails for solvable Lie algebras of dimension six. Examples of six dimensional nilpotent Lie algebras having neither left invariant symplectic nor complex structures but with generalized complex structures are given in [8].

Let us consider the solvable 6-dimensional Lie algebra $\mathfrak{g}_6$ defined by the structure equations

$$d\alpha^i = 0,\ (1 \le i \le 4), \quad d\alpha^5 = \alpha^{12} + \mu\alpha^{15}, \quad d\alpha^6 = \alpha^{15} + \alpha^{34}, \quad \mu \neq 0. \tag{30}$$

Let ω be a 2-form $\omega = \sum_{1\le i<j\le 6} \omega_{ij}\alpha^{ij}$. Then, one can check that ω is closed if and only if

$$\omega_{16} = \omega_{26} = \omega_{36} = \omega_{46} = \omega_{56} = \omega_{25} = \omega_{35} = \omega_{45} = 0,$$

that is, ω is expressed as

$$\omega = \sum_{i=2}^{5} \omega_{1i}\alpha^{1i} + \sum_{i=3}^{4} \omega_{2i}\alpha^{2i} + \omega_{34}\alpha^{34}. \tag{31}$$

But such a form ω is degenerate. This means that the Lie algebra $\mathfrak{g}_6$ does not admit generalized complex structures of type 0.

On the other hand, we consider the basis $\{X_i\}_{i=1}^6$ dual to the basis of 1-forms $\{\alpha^i\}_{i=1}^6$. From the equations (30) we get that the only non-zero Lie brackets are

$$[X_1, X_2] = -X_5, \quad [X_1, X_5] = -X_6 - \mu X_5, \quad [X_3, X_4] = -X_6.$$

Let J be an almost complex structure on $\mathfrak{g}_6$ defined by $J(X_i) = \sum_{j=1}^6 a_{ij}X_j$. Then, the components $(N_J)_{ij}^k$ of the Nijenhuis tensor N_J of J satisfy

$$(N_J)_{46}^3 = a_{63}{}^2, \quad (N_J)_{36}^4 = -a_{64}{}^2, \quad (N_J)_{56}^1 = \mu a_{51}a_{61} + a_{61}{}^2, \quad (N_J)_{26}^1 = a_{51}a_{61}.$$

Therefore, $a_{61} = a_{63} = a_{64} = 0$ since $N_J = 0$. Moreover, $J^2 = -\,\mathrm{Id}$ implies that $a_{21}{}^2 + a_{31}{}^2 + a_{41}{}^2 + a_{51}{}^2 \neq 0$, and the equations

$$(N_J)_{26}^6 = a_{65}a_{21}, \quad (N_J)_{36}^6 = a_{65}a_{31}, \quad (N_J)_{46}^6 = a_{65}a_{41}, \quad (N_J)_{56}^6 = a_{65}a_{51},$$

imply that $a_{65} = 0$. Now, from

$$(N_J)_{26}^5 = a_{62}a_{21}, \quad (N_J)_{36}^5 = a_{62}a_{31}, \quad (N_J)_{46}^5 = a_{62}a_{41}, \quad (N_J)_{56}^5 = a_{62}a_{51},$$

we obtain $a_{62} = 0$. But $-1 = (J^2)^6_6 = a_{66}{}^2$, which is not possible. This proves that $\mathfrak{g}_6$ does not admit complex structures or, equivalently, it does not admit generalized complex structures of type 3.

To describe generalized complex structures of types 1 or 2, we take the basis $\{\mathcal{X}_i\}_{i=1}^{12}$ of $T^*\mathfrak{g}$ given by

$$\mathcal{X}_i = (X_i, 0) \quad \text{and} \quad \mathcal{X}_{i+6} = (0, \alpha^i), \quad 1 \leq i \leq 6.$$

Notice that the matrices $\mathcal{J}_i$ in (18) have order 6

$$\mathcal{J}_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ -b_{12} & 0 & b_{23} & b_{24} & b_{25} & b_{26} \\ -b_{13} & -b_{23} & 0 & b_{34} & b_{35} & b_{36} \\ -b_{14} & -b_{24} & -b_{34} & 0 & b_{45} & b_{46} \\ -b_{15} & -b_{25} & -b_{35} & -b_{45} & 0 & b_{56} \\ -b_{16} & -b_{26} & -b_{36} & -b_{46} & -b_{56} & 0 \end{pmatrix}.$$

Analogous calculations to those performed in the previous sections allow us to obtain

$$\begin{aligned} &a_{13} = a_{14} = a_{16} = a_{26} = a_{36} = a_{46} = 0, \\ &b_{12} = b_{13} = b_{14} = b_{23} = b_{24} = b_{34} = 0, \\ &b_{15} = \mu b_{16}, \quad a_{15} = \mu a_{12}. \end{aligned}$$

Notice that $\det(\mathcal{J}_2) = 0$, and so $\mathfrak{g}_6$ does not admit generalized complex structures of type 0 as we already knew.

Let us consider

$$\begin{aligned} &(\mathcal{J}^2)^1_1 = -1 = a_{11}{}^2 + a_{12}(a_{21} + \mu a_{51}) - b_{16}(c_{16} + \mu c_{15}), \\ &(\mathcal{J}^2)^7_1 = 0 = -2\mu^2 a_{12} b_{16}. \end{aligned}$$

This leads us to distinguish the two following cases, that exhaust all the possibilities:

$$\text{(i)} \quad b_{16} \neq 0 \quad \text{but} \quad a_{12} = 0, \quad \text{(ii)} \quad b_{16} = 0 \quad \text{but} \quad a_{12} \neq 0.$$

In case (i), assuming that $\operatorname{rank} \mathcal{J}_2 = 2$ and the integrabiblity of the generalized almost complex structure, we get $(\mathcal{J}^2)^5_5 = (a_{55} - \mu a_{65})^2$, which is not possible. So, a necessary condition for having a generalized complex structure is that $\operatorname{rank} \mathcal{J}_2 = 4$, that is,

$$b_{25} \neq \mu b_{26}, \quad b_{35} \neq \mu b_{36} \quad \text{or} \quad b_{35} \neq \mu b_{56}. \tag{32}$$

Taking, for example, $b_{25} \neq \mu b_{26}$ we have the following generalized complex structure of type 1:

$$\begin{array}{llll} \mathcal{J}(X_1) = \alpha^6, & \mathcal{J}(X_2) = \alpha^5 - \mu\alpha^6, & \mathcal{J}(X_3) = X_4, & \mathcal{J}(X_4) = -X_3 \\ \mathcal{J}(X_5) = -\alpha^2, & \mathcal{J}(X_6) = -\alpha^1 + \mu\alpha^2, & \mathcal{J}(\alpha^1) = \mu X_5 + X_6, & \mathcal{J}(\alpha^2) = X_5, \\ \mathcal{J}(\alpha^3) = \alpha^4, & \mathcal{J}(\alpha^4) = -\alpha^3, & \mathcal{J}(\alpha^5) = -\mu X_1 - X_2, & \mathcal{J}(\alpha^6) = -X_1. \end{array}$$

Similar results are obtained for the remaining choices in (32).

In case (ii), that is, $b_{16} = 0$ and $a_{12} \neq 0$, the condition $\mathcal{J}^2 = -\operatorname{Id}$ implies that

$$0 = (\mathcal{J}^2)_1^8 = -\mu a_{12} b_{25}, \quad 0 = (\mathcal{J}^2)_1^9 = -\mu a_{12} b_{35}, \quad 0 = (\mathcal{J}^2)_1^{10} = -\mu a_{12} b_{45},$$
$$0 = (\mathcal{J}^2)_1^{12} = a_{12}(b_{26} + \mu b_{56}),$$

and then

$$b_{25} = b_{35} = b_{45} = 0, \quad \text{and} \quad b_{26} = -\mu b_{56}.$$

Therefore, $\operatorname{rank} \mathcal{J}_2 = 0$ or 2. If $\operatorname{rank} \mathcal{J}_2 = 0$, we have $-1 = (\mathcal{J}^2)_6^6 = {a_{66}}^2$, which is not possible. This means that $\mathfrak{g}_6$ does not admit generalized complex structures of type 3, as we mentioned before. So we must have $\operatorname{rank} \mathcal{J}_2 = 2$, that is,

$$b_{36} \neq 0, \quad b_{46} \neq 0 \quad \text{or} \quad b_{56} \neq 0. \tag{33}$$

Consider $b_{56} \neq 0$. Then for $b_{56} = 1$ we obtain the following generalized complex structure:

$$\begin{array}{lll} \mathcal{J}(X_1) = X_2, & \mathcal{J}(X_3) = X_4, & \mathcal{J}(X_5) = -\mu X_1 - \alpha^6, \\ \mathcal{J}(X_6) = \alpha^5, & \mathcal{J}(\alpha^1) = \alpha^2 + \mu\alpha^5, & \mathcal{J}(\alpha^2) = -\alpha^1 + \mu X_6, \\ \mathcal{J}(\alpha^3) = \alpha^4, & \mathcal{J}(\alpha^6) = -\mu X_2 + X_5. & \end{array} \tag{34}$$

Changing the basis $\{X_i, \alpha^i\}$ to $\{Y_i, \beta^i\}$ defined by

$$Y_i = X_i, \quad i = 1, 2, 3, 4, 6, \quad \text{and} \quad Y_5 = X_5 - \mu X_2,$$

(and the corresponding change between the dual basis $\{\alpha^i\}$ and $\{\beta^i\}$, resp.) the equations (34) can be written, as stated in Theorem 4.1, in the simplest form:

$$\mathcal{J}(Y_1) = Y_2, \mathcal{J}(Y_3) = Y_4, \mathcal{J}(Y_5) = -\beta^6,$$
$$\mathcal{J}(Y_6) = \beta^5, \mathcal{J}(\beta^1) = \beta^2, \mathcal{J}(\beta^3) = \beta^4,$$

that is, the generalized complex structure $\mathcal{J}$ has type 2. For the remaining choices in (33) we obtain similar results.

ACKNOWLEDGMENTS. We would like to thank the referee for useful comments, which improved this paper. This work has been partially supported through grants MCyT (Spain) Project BFM2001-3778-C03-02, MTM2005-08757-C04-02 and UPV 00127.310-E-15909/2004, ANPCyT (Argentina) Project PICT3-13557 and Fundación Antorchas (Argentina) Project 53900/3.

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